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SOLVING SUMMARIZED RICCATI EQUATION BY LINEAR MATRIX INEQUALITIES

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ABSTRACT
This paper discusses the numerical solution of the generalized algebraic Riccati equation associated with the stochastic linear-quadratic (LQ) problem. The Newton iteration, the Lyapunov iteration and the Linear Matrix Inequality (LMI) approach for finding a positive definite solution to generalized algebraic Riccati equations are discussed and compared numerically. Finally, in order to demonstrate the efficiency of the proposed algorithms, computational examples are provided and numerical effectiveness of the considered algorithms is commented.

Key words: Stochastic linear-quadratic control; Generalized algebraic Riccati equation; Positive definite solution; Linear matrix inequality.

INTRODUCTION

We consider the following optimization (LQ) problem:

$$\min \quad J_1(x_0,u(.)) = E\int_{t_0}^{\infty}(x(t)^T Qx(t) + u(t)^T Ru(t))dt \qquad (1)$$

with the restriction

$$dx(t) = [Ax(t) + Bu(t)]dt + [Gx(t) + Du(t)]dW(t), \qquad x(0) = x_0 \in R^n, \qquad (2)$$

where x (t) describes the system state, u (t) specifies the management of the system and the matrix coefficients A, B, G, D and Q, R are known. The matrices Q and R are symmetric (we view consider the real case). W (t) is a single dimension standard Vinerov process at t and W (0) = 0 and it is defined in an appropriate probability space. Solving the considered LQ problem passes through finding a positively defined solution of the algebraic equation Riccati:

$$A^T X + XA + Q + G^T XG - (XB + G^T XD)\times \qquad (3)$$

$$\times (R + D^T XD)^{-1}(XB + G^T XD)^T = 0$$

The LQ model finds wide applications in both theoretical analysis and practical aspect - when conducting empirical and theoretical economic studies; in financial modeling to analyze the investment behavior of the exchange, to choose an optimal portfolio of shares, to determine rational behavior in terms of differential games and naturally in many applications in the field of automatics and management of technical systems. Examples of economic and financial administration are

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discussed in literature in mass grata [1] Amman [2], Amman and Kendrick [3], and Amman Nyudeker [4] Enguerda [5] and others. Integral part of learning and testing of stochastic models of the type (1) - (2) is solving and finding a positive definite solution of the summarized Riccati equation (3). These issues focus the researcher's attentions and have intensified their research for the past ten years. The reason for this is the presence of a stochastic member in the description of the dynamic system (2). Its existence is determined by the matrices G and D. In the particular case G = D = 0 is obtained the well-known algebraic Riccati equation

$$A^T X + XA + Q + XBR^{-1}B^T X = 0 .$$

The last equation is considered in the assumptions that the matrices Q and R are positively identified and then the equation has a positively defined solution. In the stochastic case the matrices G and D are zero and this leads to the equation (3). The latter is found in literature under the name of summarized Riccati algebraic equation. For this equation it is allowed the condition of positive definiteness of the matrices Q and R, to be broken ie they have both positive and negative values. Such matrices are called indefinite. The requirement in this case is for the matrix $(R + D^T XD)$ to be positively determined. The decision X must satisfy the condition of positive definiteness of the matrix $(R + D^T XD)$. If $\tilde{X}$ is a positively defined solution for the summarized Riccati equation (3) with the property $R + D^T \tilde{X} D > 0$, then the optimal management of the model (1)-(2) is calculated by means of the formula $u(t) = -(R + D^T \tilde{X} D)^{-1}(\tilde{X}B + C^T \tilde{X} D)^T x(t)$ while the condition $x(t)$ is solution of the differential equation (2).

The purpose of this study is to show a different approach to solving the summarized Riccati equation (3), namely the solution of an appropriate optimization problem with a limit, expressed as a linear matrix inequality. This approach is an alternative to the classical solution of Riccati equation by iterative methods. Briefly, the approach is known as LMI approach and it is actively studied in several publications by the authors Rami, Zou and Moore [6] Rami and Zou [7] Yao, Zhang and Zou [8], Zou and Li [9]. This article will describe numerical experiments in which we compare three methods for finding positive solution of the summarized Riccati equation (3) - Newton's method, Lyapunov iterative formula and the LMI approach. Our goal is to study the numerical investigation and to compared them in relation to numerical efficiency by measuring the following indicators - reach of a certain accuracy, number of iterations and computation time. The theoretical properties of the three methods are known and published in the literature.

ITERATION FORMULAES

In order to introduce the iterative formulas of Newton's and Lyapunov method we define the following matrix function

$$\psi(X) = A^T X + XA + Q + C^T XC - (XB + C^T XD)(R + D^T XD)^{-1}(XB + C^T XD)^T ,$$

defined for positively defined matrix X. We record the summarized Riccati algebraic equation (3) in the form $\psi(X) = 0$. Under appropriate initial conditions $\psi(X)$ Dame and Hinrichsen [10] prove the of Newton's iteration formula to the only positively defined solution of the equation (3). Newtons iterativon formula is as follows

$$(A + F_{X_i})^T X_{i+1} + X_{i+1}(A + F_{X_i}) + Q + F_{X_i}^T R F_{X_i}^T \qquad (4)$$

$$+ (G + DF_{X_i})^T X_{i+1}(G + DF_{X_i}) = 0, i = 0,1,2,....,$$

Where F_X is calculated by the formula $F_X = -(R + D^T XD)^{-1}(XB + G^T XD)^T$. After changing in the last equation X_{i+1} with X_i in the expression $(G + DF_{X_i})^T X_{i+1}(G + DF_{X_i})$

we reach a different iteration formula for solving $\psi(X)=0$, which is called Lyapunov formula and looks as follows:

$$(A+F_{X_i})^T X_{i+1} + X_{i+1}(A+F_{X_i}) + Q + F_{X_i}^T R F_{X_i}^T \qquad (5)$$

$$+ (G+DF_{X_i})^T X_i (G+DF_{X_i}) = 0, i = 0,1,2,....$$

The theoretical properties of the formula (5) were investigated by Ivanov [1].
The application of LMI approach to solve the equation (3) consists in finding a positively defined matrix X, a solufron of the following optimization problem:

$$\max\ (I_m, X) \qquad (6)$$

$$\begin{pmatrix} A^T X + XA + Q + G^T XG & (XB + G^T XD) \\ (XB + G^T XD)^T & R + D^T XD \end{pmatrix} \geq 0$$

$$R + D^T XD > 0$$

$$X > 0$$

where <X, Y> means scalar multiplication of the matrices X and Y. The specific optimization task is known as the task of the semi defined programming or a task for solving linear matrix inequalities (LMI). There is a link between the LMI approach and the iterative solving of the Riccati equation (3). This relationship is derived and studied in [6.7]. One of the proven qualities is that if the equation $\psi(X)=0$ has a positive decision, then this decision is optimal for the optimization problem (Theorem 10 [9]).

NUMERICAL EXPERIMENTS

Here we will present the numerical experiments carried out to find a positively defined solution of the equation $\psi(X)=0$ through two iterative formulas of Newton (4), which for brevity is denoted (NI), and Lyapunov's (5) formula (LI) and the LMI approach by solving the problem (6).

The experiments are held on a CPU computer: Mobile DualCore Intel Pentium T2370, Motherboard: HP Compaq 6820s, RAM: 2048MB, Graphics: ATI Mobility Radeon X1350 in the programming environment of MATLAB. To calculate the solution X_{i+1} of (4) we transform a linear system with matrix block structure of the coefficient matrix, ie this matrix is a result of Kronecker multiplication of the coefficients of (4). To find X_{i+1} as a solution to (5) the MATLAB function lyap is used. In solving problem (4) the opportunities of the same computational environment are applied.

Several tests were conducted under the following conditions. Dimensionality of the matrices is denoted by m and m taking values of 5 to 15. Matrix Q is of the type $Q = diag[0,1,1,...,1,0]$ from the corresponding dimensional. Matrix R is chosen either 2x2 diagonal matrix or 3x3 diagonal matrix with negative values, ie we choose it to be negatively defined. The remaining coefficients A, B, D, G are generated by the built-in MATLAB function for pseudo-numbers following the formulas (using the terminology of MATLAB environment):

```
A=randn(m,m)/100-0.15*eye(m,m);      G=randn(m,m)/10;
B=2*randn(m,s);                      D=2*randn(m,s).
```

The variable s takes values 2 or 3 depending on the dimensionality of the matrix R. We choose accuracy tol = 10e-9 and a stop criterion $\| \psi(X_i) \|_2 \leq tol$.

The experiments are carried out as follows. For each value of m we perform 100 attempts for the methods of Newton and 100 attempdtsfor Lyapunov. To solve optimization problems we conduct 10 attempts foreach method. As a result of each attempt the number of iterations for achieving accuracy are reported ($\|\psi(X_i)\|_2 \le tol$). After a certain number of trials for each value of m we consider the following two variables "m_It" - the largest number of iterations of all performances and "av_It" - the average number of iterations for all performances (100 or 10 respectively). The results of the experiments are described in the accompanying tables **(Table 1, Table 2)**. They include the necessary time to calculate the solution in each of the methods of 100 performances. The choice of the initial matrix method of Newton and Lyapunov methods is $X_0 = 10I$.

CONCLUSION

By the numerical experiments with the described examples for finding a positively defined solution for the summarized algebraic Riccati equation, we can make the following conclusions. The two iterative formulas are faster than the method for solving optimization problem (6). Comparing the iterative formulas (4) and (5) we can say that Lyapunov method is faster and makes fewer calculations of each iterative step but reaches less accuracy, and Newton's method reaches high accuracy more quickly at the expense of execution time especially for larger values of m.

Table 1. *Results of the experiments in R=[-0.001 0; 0 -0.5].*

	LMI (6)		NI (4)		LI (5)	
m	m_It	av_It	m_It	av_It	m_It	av_It
	10 repetitions		100 repetitions		100 repetitions	
5	38	25.8	9	6.1	45	22.8
6	31	25.1	8	5.8	35	20.7
7	35	27.6	7	5.8	52	20.0
8	32	24.8	7	5.7	34	20.0
9	30	24.0	180	7.3	42	21.0
10	36	25.6	6	5.5	46	22.5
11	33	25.4	7	5.5	65	24.0
12	30	23.3	7	5.5	173	27.2
13	34	26.1	11	5.6	94	30.0
14	26	24.6	7	5.4	58	29.4
15	34	27.6	6	5.5	71	31.2
	Time in seconds for the execution of 100 iterations					
8	9.98		2.83		2.21	
10	16.78		3.27		2.60	
12	28.25		4.44		2.93	
15	53.44		10.77		4.25	

Table 2. *Results of experiments in R=[-1 0 0; 0 -1 0; 0 0 -2].*

	LMI (6)		NI (4)		LI (5)	
m	m_It	av It	m_It	av_It	m_It	av_It
	10 repetitions		100 repetitions		100 repetitions	
5	20	6.5	7	5.7	49	27.9
6	44	17.5	7	5.8	24	19.5
7	31	13.6	7	5.7	23	17.8
8	28	18.3	6	5.3	29	21.7
9	33	24.9	6	5.6	36	21.4
10	31	25.5	6	5.6	38	24.7
11	37	27.7	6	5.7	37	23.2
12	36	27.8	6	5.1	45	25.1
13	36	27.2	6	5.4	43	27.2
14	33	29.0	6	5.4	48	27.7
15	35	30.5	6	5.5	145	42.2
	Time in seconds for the execution of 100 iterations					
8	10.70		3.05		1.73	
10	19.24		2.94		1.68	
12	31.18		3.58		1.74	
15	67.33		7.53		1.82	

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