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Original Contribution

A NEW NEURAL NETWORK MODEL FOR SOLVING THE INTERVAL MAXIMUM FLOW PROBLEM

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ABSTRACT
In this paper, a new nonlinear neural network for solving the interval maximum flow problem is presented. The our nonlinear neural network is able to generate optimal solution to the interval maximum flow problem. The interval maximum flow problem in network is formulated as a special type of linear programming problem and it is solved by appropriately defined neural networks. The performance of the our neural network is demonstrated by means of illustrative example.

Key words: Nonlinear neural network, interval maximum flow problem, linear programming.

INTRODUCTION

The maximum flow problem is one of the classic combinatorial optimization problems with much application such as electrical power, traffic communication and transportation computer network. This problem is one of the most fundamental problems with a wide variety of scientific and engineering application. But now we want presents maximum flow problem when the connected arcs in a transportation network are represented as interval numbers. The problem is to find a flow of maximum interval value on a network from a source to a sink. Interval maximum flow problem like maximum flow problem is a kind of linear programming problem. The linear programming problem was first solved by Dantzig with simplex method sixty years ago. The simplex method developed by him, is still the most widely used numerical algorithm. Although the simplex method is efficient and elegant, but the modern numerical algorithms are very efficient and useful to solve maximum flow problem. The classical augmenting path method to find a maximum flow through a network was developed by Ford and Fulkerson, see Ford, et al. (1). This adaptation of the simplex method for networks is 200 - 300 times faster than the simplex method applied to general linear programs of the same dimensions. However they do not lend themselves to problems which require solution in real time. One promising approach to solve optimization problems in real time is to use the neural network approach. Researchers have proposed various dynamic solutions for constrained optimization problems. This approach was first proposed by Pyne in 1956 and developed by Dennis Rybashov, Karpinskay and others. Several new dynamic solvers using artificial neural network models have been developed; see Hopfield, et al. (2), Kennedy, et al. (3), Xia, et al. (4). The numerical algorithms are also used like genetic algorithm to solve the network problems, see Leung, et al. (5). A neural network model for interval maximum flow problem are presented in this paper. It has a much faster convergence rate. This model are based on a nonlinear dynamical system.

ARITHMETICS OF INTERVALS

All lower case letters denote real numbers and the upper case letters denote the interval numbers or the closed intervals on *R*.

$$A = [a_L, a_R] = \{a : a_L \le a \le a_R, a \in R\}$$

where a_L and a_R are the left and right limit of the interval *A* on the real line *R*, respectively. If $a_L = a_R$, then $A=[a, a]$ is a real number. Interval *A* is alternatively represented as $A = \langle m(A), w(A) \rangle$, where *m(A)* and *w(A)* are the mid- point and half-width (or simply be termed as `width') of interval *A*, i.e.,

$$m(A) = \frac{1}{2}(a_L + a_R)$$

$w(A)= \frac{1}{2}\ (a_R\ -\ a_L\ )$

Let $\ast$ = {+,-,×,÷} be a binary operation on the set of real numbers Then, $A\otimes B = \{a \ast b;\ a \in A\ ;\ b\in B\ \}$defines a binary operation on the set of closed intervals. In case of division it is assumed that $0 \notin B$.

If λ is a scalar, then

$$\lambda\,\mathrm{A} = \lambda\,[\,a_L\ ,\ a_R\,] = \begin{cases} \lambda[a_L,a_R] & if \quad \lambda \geq 0 \\ \lambda[a_R,a_L] & if \quad \lambda < 0. \end{cases}$$

The extended addition $\oplus$ and extended subtraction $\ominus$ are defined as follows:

$$A \oplus B = [\,a_L\ + b_L\ ,\ a_R\ +\ b_R\,]$$

$$A \ominus B\ =\ [\,a_L\ - b_R\ ,\ a_R\ -\ b_L\,].$$

The following equations also hold for $A \oplus B$ and $A \ominus B$:

$$m(A \oplus B) = m(A) + m(B)$$
$$m(A \ominus B) = m(A) - m(B)$$
$$w(A \oplus B) = w(A) + w(B)$$
$$w(A \ominus B) = w(A) - w(B)$$

Next we are going to propose another differentiation of interval valued. If there exists an interval C such that $A=B+C$, then C is called the *Hukuhara difference*. We also write $C = A \ominus B$ (Banks and Jacobs). Let $A = [\,a_L\,, a_R\,]$ and $B = [\,b_L\,, b_R\,]$ be two closed intervals in R. If there exists a closed interval $C = [\,c_L\,, c_R\,]$ such that $A = B + C$, then C is called the Hukuhara difference. Since $A = B + C$, it is easy to see that $a_L = b_L + c_L$ and $a_R = b_R + c_R$, i.e., $c_L = a_L - b_L$ and $c_R = a_R + b_R$. Therefore, this closed interval C exists if $a_L - b_L \leq a_R - b_R$. In this case, $C = [\,a_L - b_L\ ,\ a_R - b_R\,]$ and we also write $C = A \ominus B$. Therefore, when we say that the Hukuhara difference $C = A \ominus B$ exists, we implicitly means that

$a_L - b_L \leq a_R - b_R$.

On comparing intervals

Here we find two transitive order relations defined over intervals: the first one as an extension of < on the real line as

$$A < B\ \ iff\ \ a_R\ <\ b_L$$

and the another one as an extension of the concept of set inclusion, i.e.,

$$A \subseteq B\ iff\ a_L \geq b_L\ \ and\ a_R\ \leq b_R$$

These order relations cannot explain ranking between two overlapping intervals. The extension of the set inclusion here only describes the condition that the interval A is nested in B; but it cannot order A and B in terms of value. Ishibuchi and Tanaka approached the problem of ranking two intervals numbers more prominently.

In their approach, in a maximization problem if intervals A and B are two, say, profit intervals, then maximum of A and B can be defined by an order relation $\leq_{LR}$ between A and B as follows:

$$A \leq_{LR} B\ \ iff\ \ a_L \leq b_L\ \ and\ a_R\ \leq b_R\ ;$$

$$A <_{LR}\ B\ \ iff\ \ A\ \leq_{LR}\ B\ and\ A \neq B.$$

Ishibuchi and Tanaka suggested another order relation $\leq mw$ where $\leq_{LR}$ cannot be applied, as follows:

$$A \leq_{mw} B\ iff\ \ m(A \leq m(B)\ and\ w(A) \geq w(B);$$

$$A\ <_{mw} B\ \ iff\ \ A \leq_{mw} B\ and\ \ A \neq B.$$

The order relations $\leq_{LR}$ and $\leq_{mw}$ are antisymmetric, refexive and transitive and hence, define partial ordering between intervals. But they did not compare the pairs of intervals for which both $\leq_{LR}$ and $\leq_{mw}$ fail. As we conceive, here lies a drawback in their approach. They concentrated more on preference ordering, particularly on strict preference ordering, not on the ranking in terms of value. While considering preference ordering between two inexactness represented by two intervals, consideration of weak preference ordering between two intervals is more significant from a decision making point of view, particularly when a decision is to be made in an inexact environment. From this point of view their approach is incomplete and hence looses its significance.

PROBLEM FORMULATION

Consider a network with m nodes and n arcs that the connected arcs in a transportation network are represented as interval numbers. We associate with each arc (i, j) , a lower bound on flow of [l_{ij}^L, l_{ij}^U]=[0,0] and an upper bound on flow [u_{ij}^L, u_{ij}^U]. We shall assume throughout the development that [u_{ij}^L, u_{ij}^U] 's are finite integers interval. In such a network, we wish to find the maximum amount of flow from node 1 to node m . Let [$f^{\ L}$, $f^{\ U}$] represent the amount of flow in the network from node 1 to node m . Then the interval maximum flow problem may be stated as follows:

Maximize $[f^L, f^U]$
subject to

$$\sum_{j=1}^{m}[x_{ij}^L, x_{ij}^U] - \sum_{k=1}^{m}[x_{ki}^L, x_{ki}^U] = \begin{cases} [f^L, f^U] & i=1 \\ [0,0] & i\neq 1,m \\ [f^U, f^L] & i=m \end{cases} \tag{1}$$

$$[x_{ij}^L, x_{ij}^U] \le [u_{ij}^L, u_{ij}^U] \qquad i,j=1,2,\ldots,m$$
$$[x_{ij}^L, x_{ij}^U] \ge [0,0] \qquad i,j=1,2,\ldots,m.$$

If we use Hukuhara difference for (1), we can write (eq. 1) like as follow:

Maximize $[f^L, f^U]$
subject to

$$\sum_{j=1}^{m} x_{ij}^L - \sum_{k=1}^{m} x_{ki}^L = \begin{cases} f^L & i=1 \\ 0 & i\neq 1,m \\ f^U & i=m \end{cases} \tag{2}$$

$$\sum_{j=1}^{m} x_{ij}^U - \sum_{k=1}^{m} x_{ki}^U = \begin{cases} f^U & i=1 \\ 0 & i\neq 1,m \\ f^L & i=m \end{cases}$$

$$0 \le x_{ij}^L \le u_{ij}^L \qquad i,j=1,2,\ldots,m$$
$$0 \le x_{ij}^U \le u_{ij}^U \qquad i,j=1,2,\ldots,m.$$

If A_1 be matrix of coefficients in under m equality constraints :

$$\sum_{j=1}^{m} x_{ij}^L - \sum_{k=1}^{m} x_{ki}^L = \begin{cases} f^L & i=1 \\ 0 & i\neq 1,m \\ f^U & i=m, \end{cases}$$

also A_2 be matrix of coefficients in under m equality constraints:

$$\sum_{j=1}^{m} x_{ij}^U - \sum_{k=1}^{m} x_{ki}^U = \begin{cases} f^U & i=1 \\ 0 & i\neq 1,m \\ f^L & i=m, \end{cases}$$

and A_3, A_4 be matrixes of coefficients in under constraints respectively:

$$0 \le x_{ij}^L \le u_{ij}^L \qquad i,j=1,2,\ldots,m$$
$$0 \le x_{ij}^U \le u_{ij}^U \qquad i,j=1,2,\ldots,m.$$

So (eq.2) can be converted as follows:

$$
\begin{aligned}
&\text{Maximize} \qquad z(x)=[f^L, f^U] \\
&\text{subject to} \\
&\quad A_1 x = 0 \\
&\quad A_2 x = 0 \\
&\quad A_3 x \le b_1 \\
&\quad A_4 x \le b_2 \\
&\quad x \ge 0
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
&x = [f^L, f^U, x^L_{12}, x^U_{12}, \dots, x^L_{mm-1}, x^U_{mm-1}], && x \in R^{m(m-1)+2} \\
&b_1 = [u^L_{12}, u^L_{13}, \dots, u^L_{mm-1}], && b_1 \in R^{m(m-1)} \\
&b_2 = [u^U_{12}, u^U_{13}, \dots, u^U_{mm-1}], && b_2 \in R^{m(m-1)}.
\end{aligned}
$$

Theorem 3.1. If x^* be an optimal solution of the following problem then x^* is an optimal solution for the problem (eq.3).

$$
\begin{aligned}
&\text{Maximize} \qquad f^L + f^U \\
&\text{subject to} \\
&\sum_{j=1}^m x^L_{ij} - \sum_{k=1}^m x^L_{ki} = \begin{cases} f^L & i=1 \\ 0 & i \neq 1,m \\ f^U & i=m \end{cases} \\
&\sum_{j=1}^m x^U_{ij} - \sum_{k=1}^m x^U_{ki} = \begin{cases} f^U & i=1 \\ 0 & i \neq 1,m \\ f^L & i=m \end{cases} \\
&0 \le x^L_{ij} \le u^L_{ij} \qquad i,j=1,2,\dots,m \\
&0 \le x^U_{ij} \le u^U_{ij} \qquad i,j=1,2,\dots,m.
\end{aligned}
\tag{4}
$$

Proof. Firest problem (4) can be converted as follows:

$$
\begin{aligned}
&\text{Maximize} \qquad f^L + f^U \\
&\text{subject to} \\
&\quad A_1 x = 0 \\
&\quad A_2 x = 0 \\
&\quad A_3 x \le b_1 \\
&\quad A_4 x \le b_2 \\
&\quad x \ge 0.
\end{aligned}
\tag{5}
$$

We are going to prove this theorem by contradiction. Suppose that x^* is not a optimal solution of problem (eq.3). Then, according to Definition there exists an $x^\circ \in X$ such that $f(x^\circ) \prec_{LR} f(x^*)$, i.e.

$$(6)\quad \begin{cases} f^L < f^{*\,L} \\ f^U \le f^{*\,U} \end{cases} \quad \text{or} \quad \begin{cases} f^L \le f^{*\,L} \\ f^U < f^{*\,U} \end{cases} \quad \text{or} \quad \begin{cases} f^L < f^{*\,L} \\ f^U < f^{*\,U}. \end{cases}$$

Therefore, from above, we see that $f^L + f^U < f^{*\,L} + f^{*\,U}$ and this contradict. □

Theorem 3.2. The optimal solution of (eq.4) is equivalent to optimal solutions of the following problems.

$$\text{Maximize} \quad f^L$$
$$\text{subject to}$$
$$\sum_{j=1}^{m} x_{ij}^L - \sum_{k=1}^{m} x_{ki}^L = \begin{cases} f^L & i=1 \\ 0 & i\neq 1,m \\ f^U & i=m \end{cases} \tag{7}$$
$$0 \le x_{ij}^L \le u_{ij}^L \qquad i,j=1,2,\dots,m.$$

And

$$\text{Maximize} \quad f^U$$
$$\text{subject to}$$
$$\sum_{j=1}^{m} x_{ij}^U - \sum_{k=1}^{m} x_{ki}^U = \begin{cases} f^U & i=1 \\ 0 & i\neq 1,m \\ f^L & i=m \end{cases} \tag{8}$$
$$0 \le x_{ij}^U \le u_{ij}^U \qquad i,j=1,2,\dots,m.$$

Proof. Consider

$$C = [1,1,0,\dots,0], \qquad C \in R^{m(m-1)+2}$$
$$C_1 = [1,0,\dots,0], \qquad C_1 \in R^{m(m-1)+2}$$
$$C_2 = [0,1,0,\dots,0], \qquad C_2 \in R^{m(m-1)+2}$$

Therefore (eq.5), (eq.7), (eq.8) can be converted respectively as follow:

$$\begin{aligned} &\text{Maximize} \quad Cx = f^L + f^U \\ &\text{subject to} \\ &A_1 x = 0 \\ &A_2 x = 0 \\ &A_3 x \le b_1 \\ &A_4 x \le b_2 \\ &x \ge 0 \end{aligned} \tag{9}$$

$$\begin{aligned} &\text{Maximize} \quad C_1 x = f^L \\ &\text{subject to} \\ &A_1 x = 0 \\ &A_3 x \le b_1 \\ &x \ge 0 \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{Maximize} \quad C_2 x = f^U \\ &\text{subject to} \\ &A_2 x = 0 \end{aligned} \tag{11}$$

$A_4 x \leq b_2$

$x \geq 0$

Note that A_1, A_2, A_3, A_4 and x, b_1, b_2 be difined before.

Suppose that x^* is optimal solution for problem (eq.9), thus we can write Karush-Kuhn-Tucker optimality conditions for it. For to obtain complementary slackness condition we have:

$$w_5(-A_3\ x^* + b_1) = 0 \qquad v\,x^* = 0$$
$$w_6(-A_4\ x^* + b_2) = 0$$

Note that $w_1, w_2, w_3, w_4, w_5, w_6$ and v are Lagrangian multipliers for $A_1 x \geq 0$, $-A_1 x \geq 0$, $A_2 x \geq 0$, $-A_2 x \geq 0$, $-A_3 x \geq -b_1$, $-A_4 x \geq -b_2$, $x \geq 0$, respectively.

Also for feasible duality condition we have

$$(w_1, w_2, w_3, w_4, w_5, w_6)\begin{pmatrix} A_1 \\ -A_1 \\ A_2 \\ -A_2 \\ -A_3 \\ \end{pmatrix} + v = -C$$

Let $w_1 - w_2 = W_1$ and $w_3 - w_4 = W_2$ thus we have

$$W_1 A_1 + W_2 A_2 + w_5 A_3 + w_6 A_4 - v = C$$

The first contition of K.K.T conditions is also true. Therefore K.K.T conditions for x^* is as follow:

(i) $A_1 x^* = 0, A_2 x^* = 0, A_3 x^* \leq b_1, A_4 x^* \leq b_2$ $\qquad x^* \geq 0$

(ii) $W_1 A_1 + W_2 A_2 + w_5 A_3 + w_6 A_4 - v = C$ $\qquad W_1, W_2$ are free, $w_5, w_6, v \geq 0$

(iii) $W_1(A_1 x^* - 0) + W_2(A_2 x^* - 0) + w_5(-A_3 x^* + b_1) + w_6(-A_4 x^* + b_2) = 0$ $\qquad v\,x^* = 0.$

We should prove that K.K.T conditions for (eq.10) and (eq.11) is equivalent to K.K.T conditions above.

We can write K.K.T conditions for (eq.10) and (eq.11) like above. So K.K.T conditions for problem (eq.10) is as follow:

(i) $A_1 x^* \geq 0,\ A_3 x^* \leq b_1$ $\qquad x^* \geq 0$

(ii) $Y_1 A_1 + y_5 A_3 - v = C_1$ $\qquad Y_1$ is free $, y_5, v \geq 0$

(iii) $Y_1 (A_1 x^* - 0) + y_5 (-A_3 x^* + b_1) = 0$ $\qquad v\,x^* = 0.$

Similarrly K.K.T conditions for problem (eq.11) is as follow:

(i) $A_2 x^* \geq 0,\ A_4 x^* \leq b_2$ $\qquad x^* \geq 0$

(ii) $Y_2\ A_2 + y_6 A_4 - v = C_2$ $\qquad Y_2$ is free, $y_6, v \geq 0$

(iii) $Y_2\ (A_2 x^* - 0) + y_6(-A_4\ x^* + b_2) = 0$ $\qquad v\,x^* = 0.$

If in K.K.T conditions for two problems (eq.10) and (eq.11) we consider that $W_1 = Y_1$, $W_2 = Y_2$, $w_5 = y_5$, $w_6 = y_6$ and combination of C_1, C_2 be C thus from combination of K.K.T conditions two problems we have:

(i) $A_1 x^* \geq 0, A_2 x^* \geq 0, A_3 x^* \leq b_1, A_4 x^* \leq b_1$ $\qquad x^* \geq 0$

(ii) $Y_1 A_1 + Y_2 A_2 + y_5 A_3 + y_6 A_4 - v = C$ $\qquad Y_1, Y_2$ are free , y_5, y_6 $v \geq 0$

(iii) $Y_1 (A_1 x^* - 0) + Y_2 (A_2 x^* - 0) + y_5 (-A_3 x^* + b_1) + y_6 (-A_4 x^* + b_2) = 0$ $\qquad v x^* = 0.$

We see that K.K.T conditions for problem (eq.9) and problems (eq.10) and (eq.11) is equality. □

Result 3.1. If x^* be optimal solution of the problems (eq.7) and (eq.8) then x^* is an optimal solution of problem (eq.1).

THE NEURAL NETWORK MODEL

In this section, we use the penalty method to solve the linear programming problem (eq.10) and (eq.11) and then construct a neural network model. The first we write neural network model for (eq.10) namely for lower bound. So, if we apply penalty method to solve (eq.10), following unconstrained problem can be obtained:

$$\text{Maximize} \qquad P_1(x) = c_1 x - \frac{\mu}{2}\left[\sum_{i=1}^{n}(g_i^+ x)^2 + \sum_{j=1}^{m}(h_j x)^2\right] \tag{12}$$

wherever μ is a positive number and $g_i(x) = a_3^i x - b_1$, $h_j(x) = a_1^j x$ and $g_i^+(x) = \max\{0, g_i(x)\}$ *(i=1,2,…,n and j=1,2,…,m)*

n is number inequality constraints, m is number equality constraints, a^i is i 'th row of matrix A_2 and a^j is j 'th row of matrix A_1. So, the necessary condition for optimality of the unconstrained problem (12) is, i.e.:

$$c_1 - \mu\left[\sum_{i=1}^{n}\left(g_i^+(x)\frac{\partial g_i^+(x)}{\partial x}\right) + \sum_{j=1}^{m}\left(h_j(x)\frac{\partial h_j(x)}{\partial x}\right)\right] = 0 \qquad x \geq 0, \qquad \frac{\partial P(x)}{\partial x} = 0$$

where

$$\frac{\partial g_i(x(t))}{\partial x} = \left[\frac{\partial g_i(x(t))}{\partial f^L}, \frac{\partial g_i(x(t))}{\partial f^U}, \ldots, \frac{\partial g_i(x(t))}{\partial x^U_{m(m-1)}}\right] = a_3^i,$$

$$\frac{\partial h_j(x(t))}{\partial x} = \left[\frac{\partial h_j(x(t))}{\partial f^L}, \frac{\partial h_j(x(t))}{\partial f^U}, \ldots, \frac{\partial h_j(x(t))}{\partial x^U_{m(m-1)}}\right] = a_1^j,$$

The neural network model for the lower bound maximum flow problem (eq.10) can be described by the following nonlinear dynamical system:

$$\frac{dx}{dt} = c_1 - \mu\left[\sum_{i=1}^{n}(a_3^i)^t (a_3^i x - b_1^i)^+ + \sum_{j=1}^{m}(a_1^j)(a_1^j x)\right], \qquad x \geq 0, \tag{13}$$

Also if we apply penalty method to solve (eq.11), following unconstrained problem can be obtained:

$$\text{Maximize} \qquad P_2(x) = c_2 x - \frac{\mu}{2}\left[\sum_{i=1}^{n}(s_i^+ x)^2 + \sum_{j=1}^{m}(t_j x)^2\right] \tag{14}$$

wherever μ is a positive number and $s_i(x) = a_4^i x - b_2$, $t_j(x) = a_2^j x$ and $s_i^+(x) = \max\{0, s_i(x)\}$ *(i=1,2,…,n and j=1,2,…,m)*

n is number inequality constraints, m is number equality constraints, a^i is i 'th row of matrix A_4 and a^j is j 'th row of matrix A_2.

The neural network model for the upper bound maximum flow problem (eq.11) can be described by the following nonlinear dynamical system:

$$\frac{dx}{dt} = c_2 - \mu\left[\sum_{i=1}^{n}(a_4^i)^t(a_4^i x - b_2^i)^+ + \sum_{j=1}^{m}(a_2^j)(a_2^j x)\right], \qquad x \geq 0 \tag{15}$$

Stability analysis of the neural network model

In this part, the stability of the equilibrium state and convergence of the neural network (eq.13) to optimal solution are discussed. For nonlinear system, the most common method to show that a system is asymptotically stable is to use the Lyapunov function method, see Sontag (5), Forti, et al. (6). Assume that $v(x)=-P_1(x)$ and based on Theorem 4.1.1, $v(x)$ is as Lyapunov function and dynamical system is asymptotically stable at equilibrium state. So with respect to Theorem (4.1.2), obtain where optimal solution of maximum flow problem is equal to equilibrium state of (13).

Theorem 4. 1. 1: Under the penalty method, $v(x)$ of (eq.12) is a Lyapunov function of system (eq.13).

Proof:

$v(x)$ is a differentiable and positive definite on some neighborhood of equilibrium state, because $v(0) = 0$ and μ is an arbitrary positive number so $v(x) > 0$, for $x \neq 0$. It is sufficient for $x \neq 0$ show that $\frac{\partial(v(x(t)))}{\partial t} < 0$

For this purpose with taking the derivative of $v(x)$ with respect to time t, for $x \neq 0$ we have:

$$\frac{d(v(x(t)))}{dt} = -c_1\dot{x} - \mu\left[\sum_{i=1}^{n}(a_3^i)^t(a_3^i x - b_1^i)^+\dot{x} + \sum_{j=1}^{m}(a_1^j)(a_1^j x)\dot{x}\right] = (-\dot{x})^t\dot{x} = -\|\dot{x}\|^2 < 0.$$

Thus $v(x)$ is a Lyapunov function.

Now in Theorem 4.1.2, we prove that the optimal solution of (eq.12) is the equilibrium state of (eq.13).

Theorem 4. 1. 2: If for any μ (12) has an optimal solution, and if for system (eq.13) we can find a state variable $x(t)$, so that the neural network is asymptotically stable at x^*, then the optimal solution to (eq.12) would be the equilibrium state of (eq.13).

Proof:

The necessary condition for optimality of (eq.13) is $\frac{\partial P(x)}{\partial x} = 0$, i.e.:

$$c_1 - \mu\left(\sum_{i=1}^{n}(a_3^i)^t(a_3^i x - b_1^i)^+ + \sum_{j=1}^{m}(a_1^j)(a_1^j x)\right) = 0.$$

This is equivalent to

$$-c_1 + \mu\left(\sum_{i=1}^{n}(a_3^i)^t(a_3^i x - b_1^i)^+ + \sum_{j=1}^{m}(a_1^j)(a_1^j x)\right) = 0.$$

With regard to definition of stability in the equilibrium point we have $\frac{dx^*}{dt} = 0$.

Using Theorem 4. 1. 1, system (eq.13) is asymptotically stable, thus equilibrium state x^* satisfies (eq.12) and this lead to this fact that optimal solution of (eq.12) can be the same equilibrium state of (eq.13).

Numerical Example

Example 5. 1. Consider the maximum flow problem for following network

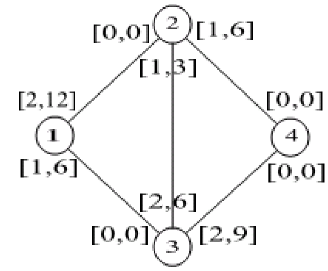

Figure 1.

First we write interval maximum flow problem like two subproblem as follows:

Maximize f^L

subject to

$$
\begin{aligned}
&x^L_{12} + x^L_{13} - x^L_{21} - x^L_{31} = f^L\\
&x^L_{21} + x^L_{23} + x^L_{24} - x^L_{21} - x^L_{32} - x^L_{42} = 0\\
&x^L_{31} + x^L_{32} + x^L_{34} - x^L_{13} - x^L_{23} - x^L_{43} = 0\\
&x^L_{42} + x^L_{43} - x^L_{24} - x^L_{34} = -f^L\\
&x^L_{12} \le 2\\
&x^L_{13} \le 1\\
&x^L_{23} \le 1\\
&x^L_{24} \le 1\\
&x^L_{32} \le 2\\
&x^L_{34} \le 2\\
&f^L, x^L_{12}, x^L_{13}, x^L_{23}, x^L_{24}, x^L_{34}, x^L_{21}, x^L_{31}, x^L_{32}, x^L_{42}, x^L_{43} \ge 0,
\end{aligned}
$$

And

Maximize f^U

subject to

$$
\begin{aligned}
&x^U_{12} + x^U_{13} - x^U_{21} - x^U_{31} = f^U\\
&x^U_{21} + x^U_{23} + x^U_{24} - x^U_{21} - x^U_{32} - x^U_{42} = 0\\
&x^U_{31} + x^U_{32} + x^U_{34} - x^U_{13} - x^U_{23} - x^U_{43} = 0\\
&x^U_{42} + x^U_{43} - x^U_{24} - x^U_{34} = -f^U\\
&x^U_{12} \le 2\\
&x^U_{13} \le 1\\
&x^U_{23} \le 1\\
&x^U_{24} \le 1\\
&x^U_{32} \le 2\\
&x^U_{34} \le 2\\
&f^U, x^U_{12}, x^U_{13}, x^U_{23}, x^U_{24}, x^U_{34}, x^U_{21}, x^U_{31}, x^U_{32}, x^U_{42}, x^U_{43} \ge 0
\end{aligned}
$$

where x^L_{ij}'s and x^U_{ij}'s are lower and upper variables respectively. Optimal solution which obtained by simplex method for first subproblem is:

$$f^{L*}=3,\quad x^{L*}_{12}=2,\quad x^{L*}_{13}=1,\quad x^{L*}_{23}=1,\quad x^{L*}_{24}=1,\quad x^{L*}_{34}=2,\quad x^{L*}_{32}=0.$$

And for second subproblem is:

$$f^{U*}=15,\quad x^{U*}_{12}=9,\quad x^{U*}_{13}=6,\quad x^{U*}_{23}=3,\quad x^{U*}_{24}=6,\quad x^{U*}_{34}=9,\quad x^{U*}_{32}=0.$$

Now we solve two subproblem above with neural network, for the first subproblem,

let $x_1 = x^L_{12},\ x_2 = x^L_{13},\ x_3 = x^L_{21},\ x_4 = x^L_{23},\ x_5 = x^L_{24},\ x_6 = x^L_{31},\ x_7 = x^L_{32},\ x_8 = x^L_{34},\ x_9 = x^L_{42},\ x_{10} = x^L_{43}$ so we have

Maximize f^L

subjet to

$$
\begin{aligned}
&h_1(x) = x_1 + x_2 - x_3 - x_6 - f^L = 0\\
&h_2(x) = x_3 + x_4 + x_5 - x_1 - x_7 - x_9 = 0\\
&h_3(x) = x_6 + x_7 + x_8 - x_2 - x_4 - x_{10} = 0
\end{aligned}
$$

$h_4\ (x)=\ x_9 + x_{10}\ {}^{L}_{43} - x_5 - x_8\ - f^L = 0$

$g_1\ (x)= x_1 - 2 \le 0$

$g_2\ (x)=\ x_2 - 1 \le\ 0$

$g_3\ (x)= x_4 - 1 \le\ 0$

$g_4\ (x)= x_5 - 1 \le\ 0$

$g_5\ (x)= x_7 - 2 \le\ 0$

$g_6\ (x)= x_8 - 2 \le\ 0$

$f^L, x_1, x_2, x_4, x_5, x_8, x_3, x_6\ , x_7, x_9, x_{10} \ge 0\ .$

The neural network model for solving the lower bound of maximum flow problem is the following nonlinear dynamical system:

$$\frac{dx}{dt} = c_1 - \mu\left[\sum_{i=1}^{6}(a_3^i)^t (a_3^i x - b_1^i)^+ + \sum_{j=1}^{4}(a_1^j)(a_1^j x)\right], \qquad x \ge 0.$$

This is equivalent to:

$$\frac{dx}{dt} = c_1 - \mu\left[\sum_{i=1}^{6} A_3{}^t (A_3 x - b_1^i) + \sum_{j=1}^{4} A_1^t (A_1 x)\right]$$

whenever b_1=[2, 1, 1, 1, 2, 2], c_1=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], x=($f^L, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$)

and $\frac{dx}{dt}$=($\frac{df}{dt}, \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{dx_5}{dt}, \frac{dx_6}{dt}, \frac{dx_7}{dt}, \frac{dx_8}{dt}, \frac{dx_9}{dt}, \frac{dx_{10}}{dt}$)

We also have

$$A_1 = \begin{pmatrix} -1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By selecting n=5000, μ = 100, dt=0.001 and x_0 =(1, 1, 1, 1, 1, 1, 1) and using Euler method for solving above neural network model, the optimal solution is obtained as follows:

$f^L =^* 3.008,\ x_{12}^{L\ *} = 2.000,\ x_{13}^{L\ *} = 1.005,\ x_{21}^{L\ *} = 0,\ x_{23}^{L\ *} = 1.003,\ x_{24}^{L\ *} = 1.005,$

$x_{31}^{L\ *} = 0\quad x_{32}^{L\ *} = 0.009\quad x_{34}^{L\ *} = 2.000\quad x_{42}^{L} =^* 0\quad x_{43}^{L\ *} = 0.$

Now for second subproblem let

$x_1 = x_{12}^U,\ x_2 = x_{13}^U,\ x_3 = x_{21}^U,\ x_4 = x_{23}^U,\ x_5 = x_{24}^U,\ x_6 = x_{31}^U,\ x_7 = x_{32}^U,\ x_8 = x_{34}^U,\ x_9 = x_{42}^U,\ x_{10} = x_{43}^U$, so we have

Maximize f^U

subjet to

$h_1\,(x)= x_1 + x_2 - x_3 - x_6\ - f^U = 0$

$h_2\ (x)=\ x_3 + x_4 + x_5 - x_1 - x_7\ - x_9 = 0$

$h_3\ (x)=\ x_6 + x_7 + x_8 - x_2 - x_4\ - x_{10} = 0$

$h_4\ (x)=\ x_9 + x_{10}\ {}^{L}_{43} - x_5 - x_8\ - f^U = 0$

$g_1\ (x)= x_1 - 12 \le 0$

$g_2\ (x)=\ x_2 - 6 \le\ 0$

$g_3\ (x)= x_4 - 3 \le\ 0$

$g_4\ (x)= x_5 - 6 \le\ 0$

$$g_5\ (x)= x_7\ \text{-}6 \le\ 0$$
$$g_6\ (x)= x_8\ \text{-}9 \le\ 0$$
$$f^U, x_1, x_2, x_4, x_5, x_8, x_3, x_6, x_7, x_9, x_{10} \ge 0$$

The neural network model for solving the lower bound of maximum flow problem is the following nonlinear dynamical system:

$$\frac{dx}{dt} = c_2 - \mu\left[\sum_{i=1}^{6}(a_4^i)^t (a_4^i x - b_2^i)^+ + \sum_{j=1}^{4}(a_2^j)(a_2^j x)\right], \qquad x \ge 0.$$

This is equivalent to:

$$\frac{dx}{dt} = c_2 - \mu\left[\sum_{i=1}^{6} A_4{}^t (A_4 x - b_2^i) + \sum_{j=1}^{4} A_2^t (A_2 x)\right]$$

Whenever b_1=(12, 6, 3, 6, 6, 9), c_1=(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), x=(f^U , x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})

and $\frac{dx}{dt}$ =($\frac{df}{dt}$ $\frac{dx_1}{dt}$ $\frac{dx_2}{dt}$ $\frac{dx_3}{dt}$ $\frac{dx_4}{dt}$ $\frac{dx_5}{dt}$ $\frac{dx_6}{dt}$ $\frac{dx_7}{dt}$ $\frac{dx_8}{dt}$ $\frac{dx_9}{dt}$ $\frac{dx_{10}}{dt}$)

We also have

$$A_2 = \begin{pmatrix} -1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By selecting n=27000, μ = 100, dt=0.001 and x_1=(3, 3, 3, 3, 3, 3) and using Euler method for solving above neural network model, the optimal solution is obtained as follows:

$$f^{U\,*}=15.008,\ x_{12}^{U\,*}=9.000,\ x_{13}^{U\,*}=6.005,\ x_{21}^{U\,*}=0,\ x_{23}^{U\,*}=3.002,\ x_{24}^{U\,*}=6.005,$$
$$x_{31}^{U\,*}=0,\ x_{32}^{U\,*}=0,\ x_{34}^{U\,*}=9.000,\ x_{42}^{U\,*}=0,\ x_{43}^{U\,*}=0.$$

CONCLUSION

This paper presents one neural network model to solve interval maximum flow problem. To obtain this model, the first original problem is transformed into an unconstrained optimization problem, then constructed a nonlinear dynamic system. The our nonlinear neural network is able to generated optimal solution to the interval maximum flow problem with a much faster convergence.

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